

SOME ASPECTS OF DIRECTED GRAPHS: PATHS AND CYCLES IN DIGRAPHS

PROBLEM SESSION

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- 1 HAMILTON CYCLES IN TOURNAMENTS
- 2 HAMILTON CYCLES IN DEGREE-CONSTRAINED DIGRAPHS
 - The Multi-Insertion Technique
 - the proof of the theorem
- 3 4-KINGS IN SEMICOMPLETE MULTIPARTITE DIGRAPHS

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DEFINITIONS

DEFINITION 1

In a digraph D , a vertex y is **reachable** from a vertex x if D has an (x, y) -path. In particular, a vertex is reachable from itself.

DEFINITION 2

D is **strong** if every vertex of D is reachable from every other vertex of D .

DEFINITION 3

A **tournament** is a digraph where there is exactly one arc between every pair of the vertices.

DEFINITION 4

A **Hamilton cycle** is a cycle that contains all vertices in D . A digraph is said to be **Hamiltonian** if it has a Hamilton cycle.

Clearly, Hamiltonian digraphs are necessarily strong.

VERTEX-PANCYCLICITY OF TOURNAMENTS

Camion proved that, for tournaments, the sufficiency is also true.

THEOREM 5 ([P. CAMION., 1959])

Every strong tournament is hamiltonian.

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Moon showed the following stronger result is also true.

DEFINITION 6

A digraph D is **vertex-pancyclic** if for every $x \in V(D)$ and every integer $k \in \{3, 4, \dots, n\}$, there exists a k -cycle through x in D .

THEOREM 7 ([J.W. MOON, 1966])

Every strong tournament is vertex-pancyclic.

THE PROOF OF THEOREM 7

- By induction on the length of the cycle. For any vertex $x \in V(D)$, we first show that there is a triangle through x in D .

THE PROOF OF THEOREM 7

- Suppose there is a $k - 1$ -cycle through x ($k \geq 4$). We show that there is a k cycle through x .

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HAMILTON CYCLES IN DEGREE-CONSTRAINED DIGRAPHS

The pair $\{x, y\}$ is dominated by a vertex z if $z \rightarrow x$ and $z \rightarrow y$; in this case we say that the pair $\{x, y\}$ is a **dominated pair**.

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THEOREM 8 ([J. BANG-JENSEN, G. GUTIN AND H. LI, 1996])

Let D be a strong digraph of order $n \geq 2$. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$ and $d(y) \geq n - 1$ or $d(x) \geq n - 1$ and $d(y) \geq n$. Then D is hamiltonian.

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THE MULTI-INSERTION TECHNIQUE

DEFINITION 9

Let $P = u_1 u_2 \dots u_s$ be a path in a digraph D and let $Q = v_1 v_2 \dots v_t$ be a path in $D - V(P)$. The path P **can be inserted into** Q if there is a subscript $i \in [t - 1]$ such that $v_i \rightarrow u_1$ and $u_s \rightarrow v_{i+1}$.

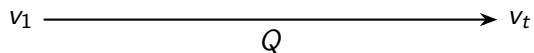
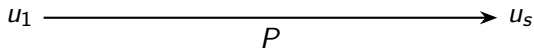
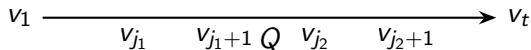
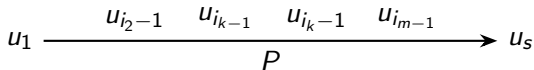


FIGURE: Inserting P into Q

THE MULTI-INSERTION TECHNIQUE

DEFINITION 10

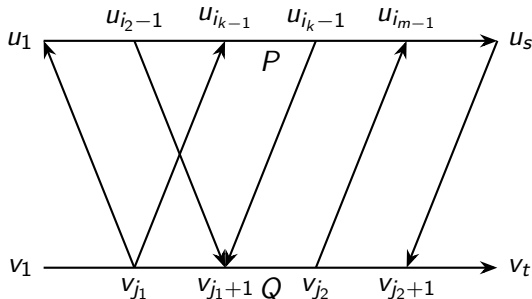
The path P can be multi-inserted into Q if there are integers $i_1 = 1 < i_2 < \dots < i_m = s + 1$ such that, for every $k = 2, 3, \dots, m$, the subpath $P[u_{i_{k-1}}, u_{i_k-1}]$ can be inserted into Q . The sequence of subpaths $P[u_{i_{k-1}}, u_{i_k-1}]$, $k = 2, \dots, m$, is a **multi-insertion partition** of P .



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THE MULTI-INSERTION TECHNIQUE

The following lemma is a simple extension of a lemma by Bang-Jensen, Gutin and Li [J. Bang-Jensen, G. Gutin and H. Li, 1996].

LEMMA 11

Let P be a path in D and let $Q = v_1v_2 \dots v_t$ be a path in $D - V(P)$. If P can be multi-inserted into Q , then there is a (v_1, v_t) -path R in D so that $V(R) = V(P) \cup V(Q)$. Given a multi-insertion partition of P , the path R can be found in time $O(|V(P)||V(Q)|)$.

A PROOF FOR THE LEMMA

PROOF.

Let $P = u_1 u_2 \dots u_s$. Suppose that integers $i_1 = 1 < i_2 < \dots < i_m = s + 1$ are such that the subpaths $P[u_{i_{k-1}}, u_{i_k}]$, $k = 2, 3, \dots, m$, form a multi-insertion partition with **the minimum** m of P . Then, every subpath can be inserted into a different arc in Q . This completes the proof of the first part.

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In the beginning, find for each subpath all arcs it can be inserted into taking at most $2(|V(Q)| - 1)$ steps. There are at most $|V(P)|$ subpaths. So this process takes $2|V(P)|(|V(Q)| - 1)$ steps.

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USEFUL LEMMA

LEMMA 12

Let $Q = v_1 v_2 \dots v_t$ be a path in D , and let $w \in V(D) - V(Q)$. If w cannot be inserted into Q , then $d_Q(w) = |N(w) \cap Q| \leq t + 1$. If, in addition, v_t does not dominate w , then $d_Q(w) \leq t$.

PROOF.



From the proof, we can see that, if we further require $w \not\rightarrow v_1$, we have $d_Q(w) \leq t - 1$. One can use the modified lemma to prove the fact that every tournament has a Hamilton path [L. Rédei, 1934].

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HAMILTON CYCLES IN DEGREE-CONSTRAINED DIGRAPHS

The pair $\{x, y\}$ is dominated by a vertex z if $z \rightarrow x$ and $z \rightarrow y$; in this case we say that the pair $\{x, y\}$ is a **dominated pair**.

THEOREM 13

([J. BANG-JENSEN, G. GUTIN AND H. LI, 1996])

Let D be a strong digraph of order $n \geq 2$. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$ and $d(y) \geq n - 1$ or $d(x) \geq n - 1$ and $d(y) \geq n$. Then D is hamiltonian.

THE PROOF OF THEOREM 13

- Assume that D is non-hamiltonian and $C = x_1x_2 \dots x_mx_1$ is a longest cycle in D . We first show that D contains a C -bypass.

THE PROOF OF THEOREM 13

- Let $P = u_1 u_2 \dots u_s$ be a C -bypass ($s \geq 3$). Without loss of generality, let $u_1 = x_1$, $u_s = x_{\gamma+1}$, $0 < \gamma < m$. Suppose also that the gap γ of P is minimum among the gaps of all C -bypasses. Since C is a longest cycle of D , $\gamma \geq 2$.

THE PROOF OF THEOREM 13

Let $R = D - V(C)$. Let x_k be an arbitrary vertex in P_1 . We first prove that

$$d(x_k) + d(u_2) \leq d_{P_2}(x_k) + 2n - |V(P_2)| - 3. \quad (1)$$

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In particular, **for any** $x_j \in P_1$ **such that** $x_1 \rightarrow x_j$. By the assumption we have

$$2n - 1 \leq d(x_j) + d(u_2) \leq d_{P_2}(x_j) + 2n - |V(P_2)| - 3$$

and therefore,

$$d_{P_2}(x_j) \geq |V(P_2)| + 2. \quad (2)$$

THE PROOF OF THEOREM 13

- By (2) and Lemma 12, x_2 can be inserted into P_2 .

THE PROOF OF THEOREM 13

- By (2) and Lemma 12, x_2 can be inserted into P_2 . Since C is a longest cycle, it follows from Lemma 11 that there exists $\beta \in \{3, \dots, \gamma\}$ so that the subpath $C[x_2, x_{\beta-1}]$ can be multi-inserted into P_2 , but $C[x_2, x_\beta]$ cannot. In particular, x_β cannot be inserted into P_2 . Now, We show that

$$d(x_\beta) \leq n - 2. \quad (3)$$

By (1), we have

$$d(x_\beta) + d(u_2) \leq d_{P_2}(x_\beta) + 2n - |V(P_2)| - 3. \quad (4)$$

Because x_β cannot be inserted into P_2 and (2), we have x_1 does not dominate x_β . By Lemma 12 we have $d_{P_2}(x_\beta) \leq |V(P_2)|$.

THE PROOF OF THEOREM 13

- Let $C[x_\alpha, x_{\beta-1}]$ ($\alpha \in \{3, \dots, \gamma\}$) be the last subpath in P_1 that can be inserted into P_2 . $x_i \rightarrow x_\alpha$ and $x_{\beta-1} \rightarrow x_{i+1}$. Observe that the pair $\{x_\beta, x_{i+1}\}$ is dominated by $x_{\beta-1}$. Thus, by (3) and the assumption of the theorem, either $x_\beta \rightarrow x_{i+1}$ or $x_{i+1} \rightarrow x_\beta$.
- $x_{i+1} \rightarrow x_\beta$.

- $x_{i+2} \rightarrow x_\beta$.

SUFFICIENT CONDITIONS FOR CONTAINING A HAMILTON PATH

DEFINITION 14

A **Hamilton path** of D is a path in D that visits all vertices.

PROPOSITION

A digraph D has a Hamilton path if and only if the digraph D^* , obtained from D by adding a new vertex x^* such that x^* dominates every vertex of D and is dominated by every vertex of D , is hamiltonian.

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Using this proposition and Theorem 13, one can prove the following sufficient condition for a digraph to have a Hamilton path.

THEOREM 15

Let D be a digraph of order n . Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n - 1$ and $d(y) \geq n - 2$ or $d(x) \geq n - 2$ and $d(y) \geq n - 1$. Then D has a Hamilton path.

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THE DISTANCE AND R-KINGS

DEFINITION 16

For a pair of vertices $u, v \in V(D)$, **the distance from u to v** , denoted by $\text{dist}(u, v)$ is the length of a shortest path from u to v

DEFINITION 17

A **r -king** is a vertex $u \in V(D)$ such that for every vertex v in D , $\text{dist}(u, v) \leq r$.

A **source** is a vertex of in-degree zero.

REMARK

For any integer r , If a digraph D has at least two sources, then it has no r -king. Thus, **we always consider digraphs with at most one source.**

2-KINGS IN TOURNAMENTS

It is easy to see every vertex of maximum out-degree in tournaments is a 2-king.

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It is easy to see every vertex of maximum out-degree in tournaments is a 2-king. Moon proved that every source-free tournament has at least three 2-kings.

THEOREM 18 ([J.W. MOON, 1962])

Every tournament with no source has at least three 2-kings.

PROOF.

For every vertex $v \in T$, there is a 2-king u that dominates v . □

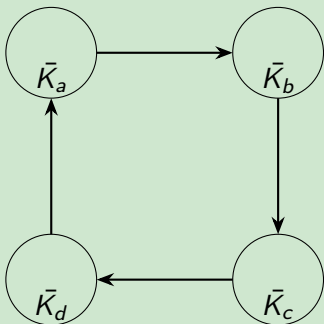
SEMICOMplete MULTIPARTITE DIGRAPHS

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DEFINITION 19

A digraph is **semicomplete multipartite** if it is obtained from a complete multipartite graph by replacing every edge by an arc or a pair of opposite arcs.

EXAMPLE 20 (A SEMICOMplete BIPARTITE DIGRAPH)

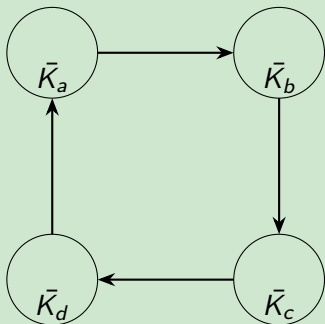


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EXAMPLE 20 (A SEMICOMplete BIPARTITE DIGRAPH)



This source-free semicomplete bipartite digraph do not have 3-kings!

THEOREM 21 ([G. GUTIN AND A. YEO, 2000])

Every semicomplete multipartite digraph with at most one source has a 4-king.

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Every semicomplete multipartite digraph with at most one source has a 4-king.

OBSERVATION

Let D be a semicomplete multipartite digraph. Then, for every adjacent pair $\{u, v\}$ and a vertex w different from them, there is at least one arc in $(\{u, v\}, w) \cup (w, \{u, v\})$.

SOME USEFUL LEMMAS

LEMMA 22

If $P = p_0 p_1 \dots p_l$ is a shortest path from p_0 to p_l in a semicomplete multipartite digraph D , and $l \geq 3$, then there is a (p_l, p_0) -path of length at most 4 in $D[V(P)]$.

PROOF.



SOME USEFUL LEMMAS

Let $N^{+i}(x) = \{y \in V(D) : \text{dist}(x, y) = i\}$ and $N^{+i}[x] = \cup_{j=0}^i N^{+j}(x)$

LEMMA 23

Let D be a strong semicomplete multipartite digraph and let w be a vertex in D . For $i \geq 3$, if $N^{+i}(w) \neq \emptyset$, then $\text{dist}(N^{+i}(w), N^{+i}[w]) \leq 4$.

PROOF.



THE PROOF OF THEOREM 21

PROOF.

- Let D be a semicomplete multipartite digraph with at most one source. If D has a vertex x of in-degree zero, then clearly x is a 2-king in D . Thus, we assume that D has no source.

THE PROOF OF THEOREM 21

PROOF.

- Let D be a semicomplete multipartite digraph with at most one source. If D has a vertex x of in-degree zero, then clearly x is a 2-king in D . Thus, we assume that D has no source.
- Then, every initial strong component Q of D has at least two vertices. Therefore, D only has one initial strong component and we denote it by Q .

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- Then, every initial strong component Q of D has at least two vertices. Therefore, D only has one initial strong component and we denote it by Q . Observe that every 4-king of Q is a 4-king of D .

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- Then, every initial strong component Q of D has at least two vertices. Therefore, D only has one initial strong component and we denote it by Q . Observe that every 4-king of Q is a 4-king of D .
- It remains to show Q has a 4-king.



Thank you for your attention!



Bang-Jensen, Gutin, and Li

Sufficient Condition for a digraph to be Hamiltonian.

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